# Mathematical Billiards and Related Topics

By Alfonso Sorrentino and Sergei Tabachnikov
Department of Mathematics, University of Rome Tor Vergata
Department of Mathematics, Pennsylvania State University

Alfonso Sorrentino

Sergei Tabachnikov

A *mathematical billiard* is a conceptually simple dynamical model describing the motion of a free particle in a domain subject to the elastic reflection off the boundary: at the impact point, the tangential component of the velocity remains the same, whereas the normal component changes the sign. As a result, the incoming and the outgoing velocity vectors have equal magnitudes and make equal angles with the boundary. This is the familiar law of geometrical optics: the angle of incidence equals the angle of reflection.

Billiards have always captured the attention of researchers in different areas of mathematics: not only is their law of motion very physical and intuitive but, due to their manifold nature, they provide a fruitful laboratory where different ideas and approaches from dynamical systems, analysis, geometry, etc..., interact and beneficially integrate.

Billiard-type dynamics is ubiquitous and there are many motivating examples for its study: Billiards appear in the analysis of many physical systems, for example: mechanical systems with elastic collisions (*i.e.*, subject to the preservation of energy and momentum) or physical systems with impacts, where a smooth bounded interaction component (*e.g.*, attraction between atoms) coexists with a short range repulsion (*e.g.*, atomic repulsion between atoms) giving rise to steep potentials, which may be approximated as a singular perturbation by elastic reflections.

Due to the above mentioned optical interpretation of the reflection law, billiard dynamics can be seen as the reflection of a ray of light inside a domain with mirrored walls, thus translating the optical properties of the domain into dynamical features of the billiard. This makes the study of billiards significant for possible applications in geometric optics (*e.g.*, freeform optical design, manufacturing of ideals lenses, etc.) We remark that optical properties of conic sections–which are related to the dynamics of the associated billiards–were well-known to the ancient Greeks. According to the legend, Archimedes used the focal property of a parabola to burn Roman ships during the Siege of Syracuse in 213–212 BCE.

A textbook example is a system of two mass-points in a segment that reflect off the endpoints and collide elastically. The configuration space of this system is 2-dimensional, and it is isomorphic to a billiard inside a right triangle whose angles depend on the ratio of the masses. See Figure 1 on the left.

A lesser-known variation is a system of three mass-points on a circle. (Figure 1 on the right.) The rotations of the circle are symmetries of this system, and after they are factored out, one obtains a billiard in an acute triangle whose angles are given by the formula

$$\alpha_i = \arctan\left(m_i\sqrt{\frac{m_1+m_2+m_3}{m_1m_2m_3}}\right), \quad i = 1, 2, 3,$$

See [13].

A much more complicated billiard system models ideal gas: the molecules are represented by balls that collide elastically. The famous Boltzmann Hypothesis of statistical physics states that this system is ergodic on a constant energy level in the phase space, and much work on billiards was done toward its proof.

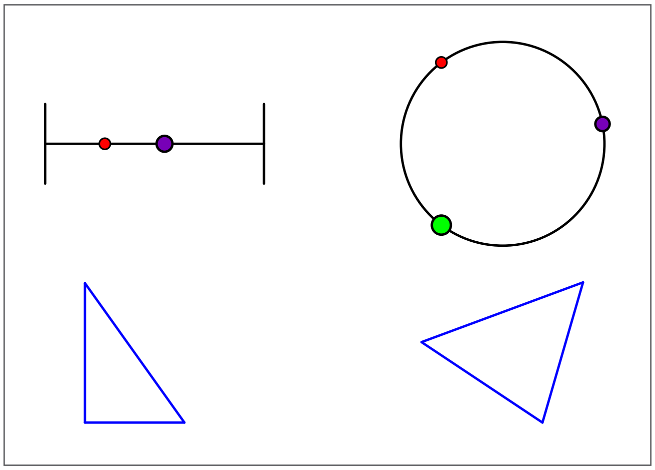

**Figure 1**: Two beads on a wire, bounded by two “walls” and three beads on a ring.

Another popular model of statistical physics is the Lorentz gas which describes the motion of electrons in metals. This is the billiard in a domain with a number of disjoint convex scatterers removed. If the domain is the plane and the scatterers are periodically positioned in identical discs, this reduces to the billiard in a torus with a circular hole, as studied by Ya. Sinai. Such billiards exhibit hyperbolic dynamical behavior.

One can consider the motion of a charge in a constant magnetic field, confined to a planar domain. The trajectories of the charge are arcs of (Larmore) circles, and the reflection off the boundary is described by the same law of geometrical optics. Such magnetic billiards (introduced by the prominent physicists Robnick and Berry [29]) are popular objects of study in mathematics and mathematical physics.

There are many other billiard-like systems and variations that can be obtained by changing the law of motion of the particle or the reflection law at the boundary. For example: outer billiards (whose study was put forward by Moser [26]), symplectic billiards (Albers and Tabachnikov [1]), wire billiards (Bialy, Mironov and Tabachnikov [7]), Finsler and Minkovski billiards (Gutkin and Tabachnikov [17]), tiling billiards, refraction billiards (De Blasi and Terracini [9], motivated by the study of certain galaxy models), etc.

The qualitative dynamical properties of billiards are extremely non-local and are profoundly intertwined with the geometry of the domain. This translates into rigidity phenomena that provide the ground for some of the foremost questions in dynamical systems, which have been challenging researchers for many decades, and have enthralling links to important open questions in other fields. Let us mention some of the most iconic questions.

*Birkhoff Conjecture*: A systematic study of billiards in strictly convex planar domains with a smooth boundary was put forward by G. Birkhoff in the early 20th century, and such billiards are named after him. The phase space of a Birkhoff billiard is comprised of the oriented chord of its boundary, and the billiard reflection defines the billiard ball transformation of this space. Topologically, this space is an annulus, and the transformation is an example of an area-preserving twist map.

An example is the billiard inside an ellipse. This dynamical system is completely integrable: every trajectory remains tangent to a confocal conic, an ellipse, or a hyperbola (or alternates between the two foci), see Figure 2. These confocal conics are caustics; they correspond to invariant curves of the billiard ball map in the phase space.

According to *Birkhoff's Conjecture*, the integrability of the billiard ball map is characteristic of ellipses. This conjecture is one of the Holy Grails of the study of mathematical billiards.

The term “integrability” can be understood in different ways: for example, one may assume that a neighborhood of the boundary of the billiard table is foliated by caustics, or that the billiard ball map possesses a polynomial integral (algebraic integrability), etc. A variety of results toward *Birkhoff's Conjecture* are available; many were obtained recently

[4, 5, 6, 15, 19, 22, 25]. We refer to [23] and references therein for a more detailed account of this topic.

*Ivrii Conjecture*: Another "big" conjecture that motivates much research in this area belongs to V. Ivrii [21]: *the periodic billiard trajectories of a billiard in Euclidean space comprise a null set.* This conjecture plays an important role in spectral analysis: assuming it, Ivrii proved the Weyl conjecture on the asymptotic behavior of the eigenvalues of the Laplace operator in a domain with the Dirichlet boundary conditions.

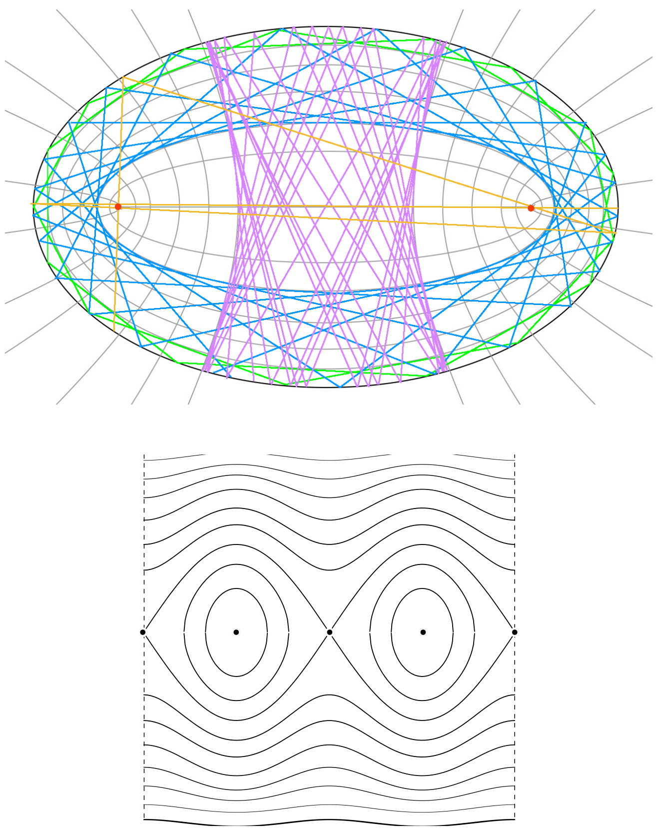

**Figure 2:** Billiard in an ellipse and its phase portrait.

Ivrii's conjecture remains open in general; however partial answers have been given. This conjecture was proven by [31] under the assumption that the boundary of the domain is analytic. A generic positive answer has been given in [27], where it is proven using a transversality theorem that for a generic domain the set of periodic orbits of any given period is finite. Note that Ivrii's conjecture holds if and only if it is true for the set of periodic orbits of any given period. Following this idea, [28] and later [30] proved the conjecture for $3$-periodic orbits. These results were also extended to $4$-periodic orbits, see [16].

*Spectral rigidity*: There is a remarkable relation between the Laplace spectrum of a domain and a dynamically defined object known as the Length Spectrum, collecting the lengths of periodic billiard trajectories in the domain. In the same spirit as Marc Kac's question "*Can one hear the shape of a drum?*", it is then natural to investigate to which extent one can "hear" a billiard domain, namely recover geometric information on its shape from its length spectrum or, possibly, show that it is spectrally determined by it, up to isometry. See recent advances [10, 11, 18].

*Symplectic capacities, Viterbo conjecture, and Mahler conjecture*: there is a surprising relation between two well-known conjectures, the Mahler conjecture regarding the volume product of symmetric convex bodies and the Viterbo conjecture, an isoperimetric-type statement for symplectic capacities of convex bodies. The length of the shortest billiard trajectory in a convex body is related to its Ekeland-Hofer-Zehnder symplectic capacity [2] and, applied to Minkowski billiards [17], this implies that the symplectic isoperimetric conjecture implies the Mahler conjecture [3].

These problems, along with many other intriguing ones, have recently experienced a renaissance thanks to significant breakthroughs. These breakthroughs have reignited research interest in these topics. It was within this context that the program titled "*Mathematical Billiards: at the Crossroads of Dynamics, Geometry, Analysis, and Mathematical Physics*" took place from October 9 to December 15, 2023, at the Simons Center for Geometry and Physics. This event provided an opportunity to gather research communities working on various aspects of the study of mathematical billiards and to explore the new and interesting directions that are emerging.

Participants were involved in a variety of activities, including weekly seminars, mini-courses, discussion groups and research projects.

Here is a list (with a short synopsis) of special mini-courses offered during the program:

Ke Zhang (University of Toronto, Canada): *KAM and formal methods for convex billiards*. In this course, an application was presented of the methods of KAM theory and formal power series to construct convex billiards with rational caustics and formally integrable billiards (after a construction of Treschev).

Vladimir Dragovic (The University of Texas at Dallas, USA): *Integrable Billiards, Extremal Polynomials, and Isoharmonic Deformations*. The aim of this course was to present a recently established synergy of integrable billiards, extremal polynomials, Riemann surfaces, combinatorics, potential theory, and isomonodromic deformations. The cross-fertilization between ideas coming from these distinct fields has been leading to new results in each of them.

Carlangelo Liverani (Università di Roma "Tor Vergata," Italy): *Statistical properties of Hyperbolic Billiards*. In this series of lectures, some recent results and open problems concerning the statistical properties of hyperbolic billiards were presented. In particular, the emphasis was put on some mathematical techniques useful to tackle such problems, such as standard pairs, dynamical functional spaces and transfer operators, strictly invariant cones, and the Hilbert metric.

Pau Martin and Rafael Ramirez-Ros (Universitat Politècnica de Catalunya, Spain): *Exponentially small phenomena in analytic convex billiards*. Many periodic and heteroclinic billiard trajectories occur in groups, so it is natural to compare their lengths. This course focused on two scenarios in which such length differences are exponentially small in some parameters. Analyticity is a key hypothesis for such phenomena. Several problems and results were described, both theoretically and numerically.

Alexey Glutsuyk (ENS Lyon, France): *On algebraically integrable planar, dual and projective billiards*. In this course, a survey was presented of the Birkhoff Conjecture and solutions by the author of its different algebraic versions and for different classes of billiard models.

The program was successful, fostering numerous engagements among participants. As a result of these many research collaborations and projects have started. Some have already resulted in preprints and more are underway. We would like to conclude this notice by describing a problem in whose study the second author of this note was involved, jointly with G. Bor and M. Spivakovsky.

Consider a Birkhoff billiard table and a source of light inside it. The rays of light that have undergone $n$ reflections off the boundary form a 1-parameter family of lines, and this family has an envelope, the $n^{\text{th}}$ *caustic by reflection*. It is known that, for every $n$, this caustic has at least four cusps, see [8]. Typically, the number of cusps grows with $n$, but computer experiments show that, if the table is elliptic, then this number remains equal to 4 for all $n$, see Figure 3.

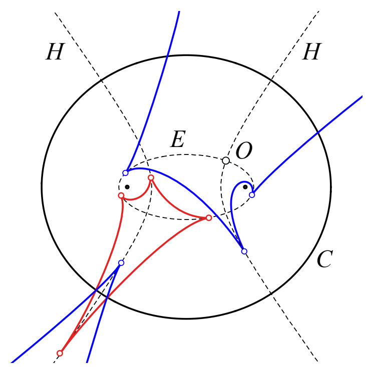


**Figure 3:** Point *O* is the source of light inside an ellipse, the red and blue curves are the first and second caustics by reflection, *E* and *H* are the confocal conics through point *O*.

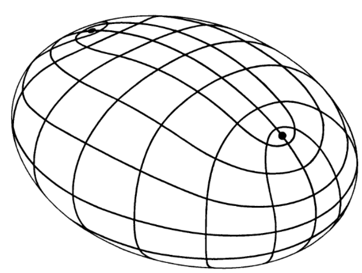

**Figure 4:** Ellipsoid with the net of the lines of curvature. The billiard inside such a curve is completely integrable (the trajectories are the geodesics, and the reflection is defined by the law of equal angles).

So far, we have proved this conjecture in the simplest case of a circle. We also know (*i.e.*, can prove it) where these four cusps are located: there are four oriented lines through point $O$ that are tangent to the two confocal conics that pass through this point, and the cusps of the $n^{\text{th}}$ caustic by reflection are the points of tangency of these $n$ times reflected rays with the respective caustics.

A similar result and a similar conjecture expand to a wider class of Liouville billiards, bounded by a coordinate curve of a Riemannian metric having a special form $(f(x)+g(y))(dx^2+dy^2)$ (see, for instance, [14]). The Euclidean plane admits elliptic coordinates in which the metric has this Liouville form and the coordinate curves are confocal conics. Another example is the surface of a triaxial ellipsoid in Euclidean space with the coordinate curves being the lines of curvature, see Figure 4.

This problem is a billiard version of a problem studied by Jacobi in his posthumously published "Lectures on Dynamics." Jacobi conjectured that the conjugate locus of a generic point on a three-axial ellipsoid had exactly four cusps. This *Last Geometric Statement of Jacobi* was proven only recently [20]. One can consider the second, thirds, etc., conjugate points along the geodesics emanating from a point, and conjecturally, each of these loci also has exactly four cusps. This problem is notoriously hard though, and not much progress has been made thus far.

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